

Suita conjecture for a punctured torus

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Abstract. For general compact Riemann surfaces, we obtain lower bounds of the Arakelov metrics. For a once-punctured complex torus, we compare the Bergman kernel and the fundamental metric, by constructing explicitly the Evans-Selberg potential and discussing its asymptotic behaviors. These works aim to generalize the Suita type results to potential-theoretically non-hyperbolic Riemann surfaces.

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1. Introduction

The Suita conjecture [13] asks about the precise relations between the Bergman kernel and the logarithmic capacity. For potential-theoretically hyperbolic Riemann surfaces, it is conjectured that the Gaussian curvature of the Suita metric is bounded from above by -4 . The relations between the Suita conjecture and the L^2 extension theorem were first observed in [10] and later contributed by several mathematicians. The Suita conjecture proves to be true for potential-theoretically hyperbolic Riemann surfaces (see [3, 6, 11]), and it might be interesting to generalize similar results to potential-theoretically non-hyperbolic cases. In this direction, for general compact Riemann surfaces we have the following result.

Theorem 1.1. *Let Y be a connected compact Riemann surface of genus $g \geq 1$. Let K , G and c be its Bergman kernel, Arakelov-Green function and Arakelov metric, respectively. In the local coordinate z , write $K = k(z)|dz|^2$ and $c = c(z)|dz|^2$. Finally let $\sup_{(p,q) \in Y \times Y} G(p,q) = s$. Then, for any $w \in Y$, it follows*

that

$$\frac{\pi k(w)}{c^2(w)} \geq \exp(-2s). \quad (1.1)$$

On the one hand, this result indicates an upper bound of the Arakelov metric's Gaussian curvature. For the genus one case the author in [4] computed via elliptic functions the above left hand side and obtained a sharp positive lower bound which is less than 1. On the other hand, for a once-punctured complex torus which is a typical example of potential-theoretically parabolic Riemann surfaces, we construct the so-called Evans-Selberg potential and therefore derive the fundamental metric.

Theorem 1.2. *Let $X_{\tau,u} := X_{\tau} \setminus \{u\}$ be a once-punctured complex torus. Let $K_{\tau,u}$ and $c_{\tau,u}$ be its Bergman kernel and fundamental metric, respectively. In the local coordinate w , write $K_{\tau,u} = k_{\tau,u}(w)|dw|^2$ and $c_{\tau,u} = c_{\tau,u}(w)|dw|^2$. Then, as $w \rightarrow u$, it follows that*

$$\frac{\pi k_{\tau,u}(w)}{c_{\tau,u}^2(w)} \sim \frac{\pi \cdot |w - u|^2}{2 \cdot \operatorname{Im} \tau} \rightarrow 0^+.$$

Moreover, we study the degenerate case when a once-punctured complex torus becomes a singular curve and obtain the following result.

Theorem 1.3. *Under the same assumptions as in Theorem 1.2, as $\operatorname{Im} \tau \rightarrow +\infty$, it follows that*

$$\frac{\pi K_{\tau,u}(w)}{c_{\tau,u}^2(w)} \rightarrow 0^+.$$

Either Theorem 1.2 or Theorem 1.3 implies that the Gaussian curvature the of fundamental metric on $X_{\tau,u}$ can be arbitrarily close to 0^- , which is different from the potential-theoretically hyperbolic case.

Corollary 1.4. *The Gaussian curvature of the fundamental metric on a once-punctured complex torus cannot be bounded from above by a negative constant.*

2. Preliminaries

Let Y be a connected compact Riemann surface of genus $g \geq 1$ whose Bergman kernel in the local coordinate z is written as $K = k(z)|dz|^2$. Then the Arakelov-Green function $g_w(z)$ on Y with a pole w satisfies the equation [5]

$$\frac{\partial^2 g_w(z)}{\partial z \partial \bar{z}} = \frac{\pi}{2} \left(\delta(z - w) - \frac{k(z)}{g} \right). \quad (2.1)$$

Definition 2.1. With the above notations, the Arakelov metric on Y under the local coordinate w is defined as

$$c(w)|dw|^2 := \exp \lim_{z \rightarrow w} (g_w(z) - \log |z - w|) |dw|^2.$$

The Arakelov metric has a characterizing property that its Gaussian curvature form is proportional to the Bergman kernel [1, 8]. Next, we recall the definition of the so-called fundamental metric, which is a non-compact counterpart of the Arakelov metric. For a potential-theoretically hyperbolic Riemann surface, the fundamental metric is just the Suita metric.

Definition 2.2. On a potential-theoretically parabolic Riemann surface X , the fundamental metric under the local coordinate z is defined as

$$c(z)|dz|^2 := \exp \lim_{w \rightarrow z} (E_w(z) - \log |z - w|) |dz|^2,$$

where $E_w(z)$ is an Evans-Selberg potential on X with a pole w .

For general potential-theoretically parabolic Riemann surfaces, it is known (see [9]) that the Gaussian curvature form of the fundamental metric is

$$-4 \frac{\partial^2}{\partial w \partial \bar{w}} \log c(z) = -4\pi k(z) (\leq 0), \quad (2.2)$$

where $k(z)$ is the coefficient of the Bergman kernel $(1, 1)$ -form under the local coordinate z . For a compact complex torus $X_\tau := \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$, where $\tau \in \mathbb{C}$ and $\text{Im } \tau > 0$, its Bergman kernel by definition is $K_\tau(z) = (\text{Im } \tau)^{-1} dz \wedge d\bar{z}$, where z is the local coordinate induced from the complex plane \mathbb{C} .

Definition 2.3. A once-punctured complex torus $X_{\tau,u} := X_\tau \setminus \{u\}$ is an open Riemann surface obtained by removing one single point u from a compact complex torus X_τ .

3. The compact case

Example (A complex torus). The Arakelov-Green function on a complex torus X_τ defined as above with a pole w satisfies the equation

$$\frac{\partial^2 g_w(z)}{\partial z \partial \bar{z}} = \frac{\pi}{2} \left(\delta(z - w) - \frac{1}{\text{Im } \tau} \right), \quad (3.1)$$

and can be expressed via the theta function as

$$g_w(z) = \log \left| \frac{\theta_1(z - w; q)}{\eta(\tau)} \right| - \frac{\pi \cdot (\text{Im}(z - w))^2}{\text{Im } \tau}. \quad (3.2)$$

Here $\eta(\tau) = q^{\frac{1}{12}} \cdot \prod_{m=1}^{\infty} (1 - q^{2m})$ and

$$\theta_1(z; q) := 2q^{1/4} \sin(\pi z) \prod_{n=1}^{\infty} (1 - q^{2n})(1 - 2 \cos(2\pi z)q^{2n} + q^{4n}),$$

for $q = \exp(\pi i \tau)$. In this case, it is possible to compare the Bergman kernel and the Arakelov metric. Instead of using programming to numerically approximate the lower bound as in [4]¹, the Gaussian curvature of the Arakelov metric can be computed by *Mathematica* (Version 10.3).

¹The author apologizes for several mistakes contained in [4].

Proof of Theorem 1.1. The argument follows almost the same as Berndtsson-Lempert's method [2] by taking sub-level sets of the Arakelov-Green function and consider variations of the Bergman kernels. Specifically, for any fixed $w \in Y$ consider $Y_t := \{p \in Y | G(p, w) < t, t \in \mathbb{R}\}$ and denote the Bergman kernel on Y_t by $K_t = k_t(w)|dw|^2$ near the pole w . Using the semipositivity properties of the direct image bundles and combining the asymptotics at $-\infty$, one gets that $\log k_t(w) + 2t$ is a convex increasing function in t and then

$$\frac{\pi k_t(w)}{c^2(w)} \geq \exp(-2t), \quad (3.3)$$

holds for all t . In particular, (3.3) is true for $t = s$ and $k_s(w) = k(w)$. \square

In general, the right hand side of (1.1) can be smaller than 1, if $s > 0$, i.e. the Arakelov-Green function can be positive at some cases. This is different from the potential-theoretically hyperbolic case where the Green function is always less than 0 in the interior. For the same reason, it is not clear how to apply Berndtsson-Lempert's method to the potential-theoretically parabolic case, since an Evans-Selberg potential tends to $+\infty$ near the boundary.

Recently, the author was kindly notified by Guan that the Bergman kernel for pluri-canonical bundles and the Arakelov metric on compact Riemann surfaces could be compared by L^2 extension theorems (see [7]).

4. A once-punctured torus

As a typical example of potential-theoretically parabolic Riemann surfaces, $X_{\tau,u}$ admits an Evans-Selberg potential and we have indeed constructed it.

Lemma 4.1. *There exists an Evans-Selberg potential on $X_{\tau,u} := X_\tau \setminus \{u\}$ with a pole w given by*

$$E_w^{\tau,u}(z) = \log \left| \frac{\theta_1(z-w; q)}{\theta_1(z-u; q)} \right|,$$

for $z \in X_{\tau,u} \setminus \{w\}$.

Proof. We see that the two terms on the right hand side of (3.2) are responsible for the two terms on the right hand side of (3.1), respectively. Keeping this in mind, we can construct the Evans-Selberg potential by attaching physics meanings. We regard the potential as an electric flux generated at the pole w and terminates at the boundary point u (see [12] for detailed physics explanations). Therefore, the Evans-Selberg potential $E_w^{\tau,u}(z)$ with a pole w satisfies that

$$\frac{\partial^2 E_w^{\tau,u}(z)}{\partial z \partial \bar{z}} = \frac{\pi}{2} (\delta(z-w) - \delta(z-u)),$$

and can be expressed via the theta function as

$$\begin{aligned}
 E_w^{\tau,u}(z) &= \log \left| \frac{\theta_1(z-w; q)}{\eta(\tau)} \right| - \log \left| \frac{\theta_1(z-u; q)}{\eta(\tau)} \right| \\
 &= \log \left| \frac{\theta_1(z-w; q)}{\theta_1(z-u; q)} \right| \\
 &= \log \left| \frac{\sin(\pi(z-w)) \cdot \prod_{m=1}^{\infty} (1 - 2 \cos(2\pi(z-w)) \cdot q^{2m} + q^{4m})}{\sin(\pi(z-u)) \cdot \prod_{m=1}^{\infty} (1 - 2 \cos(2\pi(z-u)) \cdot q^{2m} + q^{4m})} \right|. \quad \square
 \end{aligned}$$

Theorem 4.2. *There exists a fundamental metric $c_{\tau,u}$ on $X_{\tau,u} := X_{\tau} \setminus \{u\}$ under the local coordinate w given by*

$$c_{\tau,u}(w) |dw|^2 = \frac{2\pi \cdot |\eta(\tau)|^3}{|\theta_1(w-u; q)|} |dw|^2.$$

Proof. This can be verified by definition, since

$$\begin{aligned}
 c_{\tau,u}(w) &= \exp \lim_{z \rightarrow w} \left(\log \left| \frac{\theta_1(z-w; q)}{\theta_1(z-u; q)} \right| - \log |z-w| \right) \\
 &= \left| \frac{\pi \cdot \prod_{m=1}^{\infty} (1 - q^{2m})^2}{\sin(\pi(w-u)) \cdot \prod_{m=1}^{\infty} (1 - 2 \cos(2\pi(w-u)) \cdot q^{2m} + q^{4m})} \right| \\
 &= \left| \frac{\pi \cdot \eta(\tau)^2}{q^{\frac{1}{6}} \cdot \sin(\pi(w-u)) \cdot \prod_{m=1}^{\infty} (1 - 2 \cos(2\pi(w-u)) \cdot q^{2m} + q^{4m})} \right| \\
 &= \left| \frac{\pi \cdot \eta(\tau)^2 \cdot 2q^{\frac{1}{4}} \cdot \prod_{m=1}^{\infty} (1 - q^{2m})}{q^{\frac{1}{6}} \cdot \theta_1(w-u; q)} \right| \\
 &= \left| \frac{\pi \cdot \eta(\tau)^2 \cdot 2q^{\frac{1}{4}} \cdot \frac{\eta(\tau)}{q^{\frac{1}{12}}}}{q^{\frac{1}{6}} \cdot \theta_1(w-u; q)} \right| \\
 &= \left| \frac{2\pi \cdot \eta(\tau)^3}{\theta_1(w-u; q)} \right|. \quad \square
 \end{aligned}$$

By the second equality above, $c_{\tau,u}$ has the following asymptotic behavior, which will yield Theorem 1.2 for any fixed τ .

Corollary 4.3. *Under the same assumptions as in Theorem 4.2, as $w \rightarrow u$, it follows that*

$$c_{\tau,u}(w) \sim \frac{1}{|w-u|} \rightarrow +\infty.$$

5. The degenerate case

By studying the asymptotic behaviors of the fundamental metric under degeneration with respect to the complex structure, we will prove Theorem 1.3. Relating Theorem 1.3 with (2.2), we further get Corollary 1.4.

Proof of Theorem 1.3. Let $\text{Im } \tau \rightarrow +\infty$, and $q \equiv \exp(\pi i \tau)$ will tend to 0. Then, it has

$$\begin{aligned} c_{\tau,u}(w) &\rightarrow \left| \frac{\pi \cdot \prod_{m=1}^{\infty} (1 - 0^{2m})^2}{\sin(\pi(w-u)) \cdot \prod_{m=1}^{\infty} (1 - 2 \cos(2\pi(w-u)) \cdot 0^{2m} + 0^{4m})} \right| \\ &\rightarrow \frac{\pi}{|\sin(\pi(w-u))|}. \end{aligned}$$

Therefore, it follows that

$$\frac{\pi K_{\tau,u}(w)}{c_{\tau,u}^2(w)} \rightarrow \frac{|\sin(\pi(w-u))|^2}{2 \cdot \text{Im } \tau \cdot \pi} \rightarrow 0^+,$$

since the denominator is uniformly bounded by 1 for any fixed w . \square

On the one hand, at the degenerate case of potential-theoretically hyperbolic Riemann surfaces, we are not sure whether Gaussian curvatures of the Suita metrics are still bounded from above by -4 . On the other hand, for a compact complex torus, the Gaussian curvature of the Arakelov metric is always 0 by the genus reason, although our earlier result in [4] shows that as $\text{Im } \tau \rightarrow +\infty$, it has

$$\frac{\pi K_{\tau}(w)}{c_{\tau}^2(w)} \rightarrow +\infty.$$

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